

# Higher-Order Triangular-Distance Delaunay Graphs: Graph-Theoretical Properties

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## Abstract

We consider an extension of the triangular-distance Delaunay graphs (TD-Delaunay) on a set  $P$  of points in the plane. In TD-Delaunay, the convex distance is defined by a fixed-oriented equilateral triangle  $\nabla$ , and there is an edge between two points in  $P$  if and only if there is an empty homothet of  $\nabla$  having the two points on its boundary. We consider higher-order triangular-distance Delaunay graphs, namely  $k$ -TD, which contains an edge between two points if the interior of the homothet of  $\nabla$  having the two points on its boundary contains at most  $k$  points of  $P$ . We consider the connectivity, Hamiltonicity and perfect-matching admissibility of  $k$ -TD. Finally we consider the problem of blocking the edges of  $k$ -TD.

## 1 Introduction

The *triangular-distance Delaunay graph* of a point set  $P$  in the plane, TD-Delaunay for short, was introduced by Chew [12]. A TD-Delaunay is a graph whose convex distance function is defined by a fixed-oriented equilateral triangle. Let  $\nabla$  be a downward equilateral triangle whose barycenter is the origin and one of its vertices is on negative  $y$ -axis. A *homothet* of  $\nabla$  is obtained by scaling  $\nabla$  with respect to the origin by some factor  $\mu \geq 0$ , followed by a translation to a point  $b$  in the plane:  $b + \mu\nabla = \{b + \mu a : a \in \nabla\}$ . In the TD-Delaunay graph of  $P$ , there is a straight-line edge between two points  $p$  and  $q$  if and only if there exists a homothet of  $\nabla$  having  $p$  and  $q$  on its boundary and whose interior does not contain any point of  $P$ . In other words,  $(p, q)$  is an edge of TD-Delaunay graph if and only if there exists an empty downward equilateral triangle having  $p$  and  $q$  on its boundary. In this case, we say that the edge  $(p, q)$  has the *empty triangle property*. The TD-Delaunay graph is a planar graph, see [7]. We define  $t(p, q)$  as the smallest homothet of  $\nabla$  having  $p$  and  $q$  on its boundary. See Figure 1(a). Note that  $t(p, q)$  has one of  $p$  and  $q$  at a vertex, and the other one on the opposite side. Thus,

**Observation 1.** *Each side of  $t(p, q)$  contains either  $p$  or  $q$ .*

In [4], the authors proved a tight lower bound of  $\lceil \frac{n-1}{3} \rceil$  on the size of a maximum matching in a TD-Delaunay graph. In this paper we study higher-order TD-Delaunay graphs. An *order- $k$  TD-Delaunay graph* of a point set  $P$ , denoted by  $k$ -TD, is a geometric graph which has an edge  $(p, q)$  iff the interior of  $t(p, q)$  contains at most  $k$  points of  $P$ ; see Figure 1(b). The standard TD-Delaunay graph corresponds to 0-TD. We consider graph-theoretic properties of higher-order TD-Delaunay graphs, such as connectivity, Hamiltonicity, and perfect-matching admissibility. We also consider the problem of blocking TD-Delaunay graphs.

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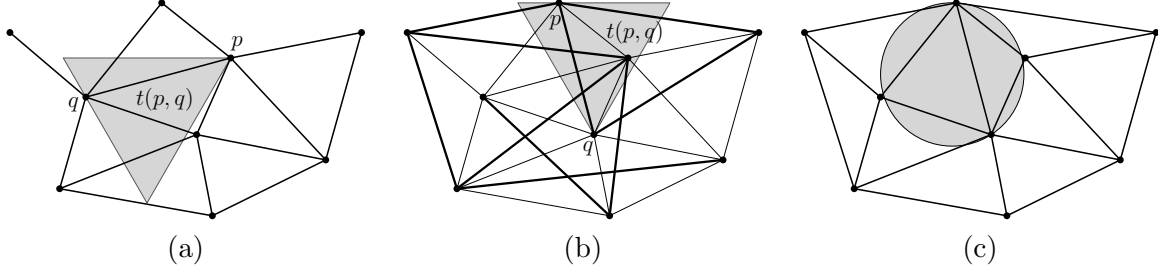


Figure 1: (a) Triangular-distance Delaunay graph (0-TD), (b) 1-TD graph, the light edges belong to 0-TD as well, and (c) Delaunay triangulation.

## 1.1 Previous Work

A *Delaunay triangulation* (DT) of  $P$  is a graph whose distance function is defined by a fixed circle  $\odot$  centered at the origin. DT has an edge between two points  $p$  and  $q$  if there exists a homothet of  $\odot$  having  $p$  and  $q$  on its boundary and whose interior does not contain any point of  $P$ ; see Figure 1(c). In this case the edge  $(p, q)$  is said to have the *empty circle property*. An *order- $k$  Delaunay Graph* on  $P$ , denoted by  $k$ -DG, is defined to have an edge  $(p, q)$  iff there exists a homothet of  $\odot$  having  $p$  and  $q$  on its boundary and whose interior contains at most  $k$  points of  $P$ . The standard Delaunay triangulation corresponds to 0-DG.

For each pair of points  $p, q \in P$  let  $D[p, q]$  be the closed disk having  $pq$  as diameter. A *Gabriel Graph* on  $P$  is a geometric graph which has an edge between two points  $p$  and  $q$  iff  $D[p, q]$  does not contain any point of  $P \setminus \{p, q\}$ . An *order- $k$  Gabriel Graph* on  $P$ , denoted by  $k$ -GG, is defined to have an edge  $(p, q)$  iff  $D[p, q]$  contains at most  $k$  points of  $P \setminus \{p, q\}$ .

For each pair of points  $p, q \in P$ , let  $L(p, q)$  be the intersection of the two open disks with radius  $|pq|$  centered at  $p$  and  $q$ . A *Relative Neighborhood Graph* on  $P$  is a geometric graph which has an edge between two points  $p$  and  $q$  iff  $L(p, q)$  does not contain any point of  $P$ . An *order- $k$  Relative Neighborhood Graph* on  $P$ , denoted by  $k$ -RNG, is defined to have an edge  $(p, q)$  iff  $L(p, q)$  contains at most  $k$  points of  $P$ . It is obvious that  $k$ -RNG  $\subseteq k$ -GG  $\subseteq k$ -DG.

The problem of determining whether an order- $k$  geometric graph always has a (bottleneck) perfect matching or a (bottleneck) Hamiltonian cycle is quite of interest. We will define these notions in Section 2.2. Chang et al. [10, 11, 9] proved that a Euclidean bottleneck biconnected spanning graph, bottleneck perfect matching, and bottleneck Hamiltonian cycle of  $P$  are contained in 1-RNG, 16-RNG, 19-RNG, respectively. This implies that 16-RNG has a perfect matching and 19-RNG is Hamiltonian. Since  $k$ -RNG is a subgraph of  $k$ -GG, the same results hold for 16-GG and 19-GG. It is known that  $k$ -GG is  $(k + 1)$ -connected [8] and 15-GG (and hence 15-DG) is Hamiltonian. Dillencourt showed that a Delaunay triangulation (0-DG) admits a perfect matching [14] but it can fail to be Hamiltonian [13].

Given a geometric graph  $G(P)$  on a set  $P$  of  $n$  points, we say that a set  $K$  of points blocks  $G(P)$  if in  $G(P \cup K)$  there is no edge connecting two points in  $P$ . Actually  $P$  is an independent set in  $G(P \cup K)$ . Aichholzer et al. [2] considered the problem of blocking the Delaunay triangulation (i.e. 0-DG) for  $P$  in general position. They show that  $\frac{3n}{2}$  points are sufficient to block DT( $P$ ) and at least  $n - 1$  points are necessary. To block a Gabriel graph,  $n - 1$  points are sufficient [3].

In a companion paper, we considered the matching and blocking problems in higher-order Gabriel graphs. We showed that 10-GG contains a Euclidean bottleneck matching and 8-GG may not have any. As for maximum matching, we proved a tight lower bound of  $\frac{n-1}{4}$  in 0-GG. We also showed that 1-GG has a matching of size at least  $\frac{2(n-1)}{5}$  and 2-GG has a perfect matching (when  $n$  is even). In addition, we showed that  $\lceil \frac{n-1}{3} \rceil$  points are necessary to block

0-TD and this bound is tight.

## 1.2 Our Results

We show for which values of  $k$ ,  $k$ -TD contains a bottleneck biconnected spanning graph, bottleneck Hamiltonian cycle, and (bottleneck) perfect-matching. We define these notions Section 2.2. In Section 3 we prove that every  $k$ -TD graph is  $(k + 1)$ -connected. In addition we show that a bottleneck biconnected spanning graph of  $P$  is contained in 1-TD. Using a similar approach as in [1, 9], in Section 4 we show that a bottleneck Hamiltonian cycle of  $P$  is contained in 7-TD. We also show a configuration of a point set  $P$  such that 5-TD fails to have a bottleneck Hamiltonian cycle. In Section 5 we prove that a bottleneck perfect matching of  $P$  is contained in 6-TD, and we show that for some point set  $P$ , 5-TD does not have a bottleneck perfect matching. In Section 5.2 we prove that 2-TD has a perfect matching and 1-TD has a matching of size at least  $\frac{2(n-1)}{5}$ . In Section 6 we consider the problem of blocking  $k$ -TD. We show that at least  $\lceil \frac{n-1}{2} \rceil$  points are necessary and  $n - 1$  points are sufficient to block a 0-TD. The open problems and concluding remarks are presented in Section 7.

## 2 Preliminaries

### 2.1 Some Geometric Notions

Bonichon et al. [6] showed that a half- $\Theta_6$  graph of a point set  $P$  in the plane is equal to a TD-Delaunay graph of  $P$ . They also showed that every plane triangulation is TD-Delaunay realizable.

A half- $\Theta_6$  graph (or equivalently a TD-Delaunay graph) on a point set  $P$  can be constructed in the following way. For each point  $p$  in  $P$ , let  $l_p$  be the horizontal line through  $p$ . Define  $l_p^\gamma$  as the line obtained by rotating  $l_p$  by  $\gamma$ -degrees in counter-clockwise direction around  $p$ . Actually  $l_p^0 = l_p$ . Consider three lines  $l_p^0$ ,  $l_p^{60}$ , and  $l_p^{120}$  which partition the plane into six disjoint cones with apex  $p$ . Let  $C_p^1, \dots, C_p^6$  be the cones in counter-clockwise order around  $p$  as shown in Figure 2. We partition the cones into the set of *odd cones*  $\{C_p^1, C_p^3, C_p^5\}$ , and the set of *even cones*  $\{C_p^2, C_p^4, C_p^6\}$ . For each even cone  $C_p^i$  connect  $p$  to the “nearest” point  $q$  in  $C_p^i$ . The *distance* between  $p$  and  $q$ ,  $d(p, q)$ , is defined as the Euclidean distance between  $p$  and the orthogonal projection of  $q$  onto the bisector of  $C_p^i$ . See Figure 2. The resulting graph is the half- $\Theta_6$  graph which is defined by even cones [6]. Moreover, the resulting graph is the TD-Delaunay graph defined with respect to homothets of  $\nabla$ .

By considering the odd cones, another half- $\Theta_6$  graph is obtained. The well-known  $\Theta_6$  graph is the union of half- $\Theta_6$  graphs defined by odd and even cones. To construct  $k$ -TD, for each point  $p \in P$  we connect  $p$  to its  $(k + 1)$  nearest neighbors in each even cone around  $p$ . It is obvious that  $k$ -TD has  $O(kn)$  edges. The  $k$ -TD can be constructed in  $O(n \log n + kn \log \log n)$ -time, using the algorithm introduced by Lukovszki [15] for computing fault tolerant spanners.

Recall that  $t(p, q)$  is the smallest homothet of  $\nabla$  having  $p$  and  $q$  on its boundary. In other words,  $t(p, q)$  is the smallest downward equilateral triangle through  $p$  and  $q$ . Similarly we define  $t'(p, q)$  as the smallest upward equilateral triangle having  $p$  and  $q$  on its boundary. It is obvious that the even cones correspond to downward triangles and odd cones correspond to upward

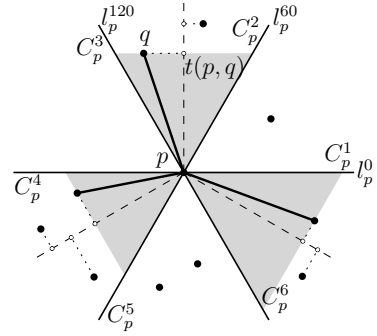


Figure 2: The construction of the TD-Delaunay graph.

triangles. We define an order on the equilateral triangles: for each two equilateral triangles  $t_1$  and  $t_2$  we say that  $t_1 < t_2$  if the area of  $t_1$  is less than the area of  $t_2$ . Since the area of  $t(p, q)$  is directly related to  $d(p, q)$ ,

$$d(p, q) < d(r, s) \quad \text{if and only if} \quad t(p, q) < t(r, s).$$

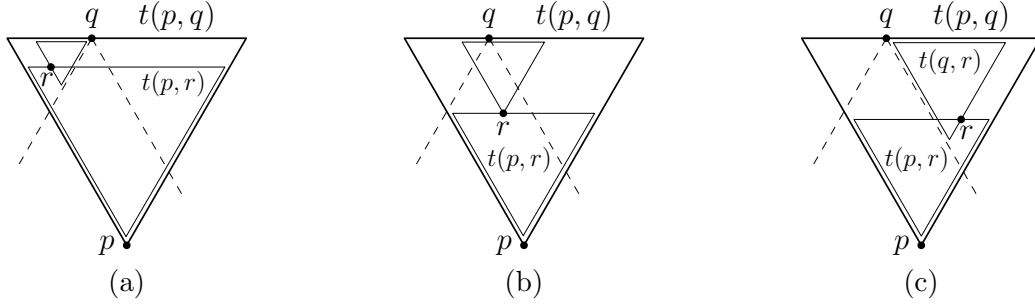


Figure 3: Illustration of Observation 2: the point  $r$  is contained in  $t(p, q)$ . The triangles  $t(p, r)$  and  $t(q, r)$  are inside  $t(p, q)$ .

As shown in Figure 3 we have the following observation:

**Observation 2.** *If  $t(p, q)$  contains a point  $r$ , then  $t(p, r)$  and  $t(q, r)$  are contained in  $t(p, q)$ .*

As a direct consequence of Observation 2, if a point  $r$  is contained in  $t(p, q)$ , then  $\max\{t(p, r), t(q, r)\} < t(p, q)$ . It is obvious that,

**Observation 3.** *For each two points  $p, q \in P$ ,  $t(p, q) = t'(p, q)$ .*

Thus, we define  $X(p, q)$  as a regular hexagon centred at  $p$  which has  $q$  on its boundary, and its sides are parallel to  $l_p^0$ ,  $l_p^{60}$ , and  $l_p^{120}$ .

**Observation 4.** *If  $X(p, q)$  contains a point  $r$ , then  $t(p, r) < t(p, q)$ .*

For each edge  $(p, q)$  in  $k$ -TD we define its *weight*,  $w(p, q)$ , to be equal to the area of  $t(p, q)$ .

## 2.2 Some Graph-Theoretic Notions

A graph  $G$  is *connected* if there is a path between any pair of vertices in  $G$ . Moreover,  $G$  is *k-connected* if there does not exist a set of at most  $k - 1$  vertices whose removal disconnects  $G$ . In case  $k = 2$ ,  $G$  is called *biconnected*. In other words a graph  $G$  is biconnected iff there is a simple cycle between any pair of its vertices. A *matching* in  $G$  is a set of edges in  $G$  without common vertices. A *perfect matching* is a matching which matches all the vertices of  $G$ . A *Hamiltonian cycle* in  $G$  is a cycle (i.e., closed loop) through  $G$  that visits each vertex of  $G$  exactly once. In case that  $G$  is an edge-weighted graph, a *bottleneck matching* (resp. *bottleneck Hamiltonian cycle*) is defined to be a perfect matching (resp. Hamiltonian cycle) in  $G$  with the weight of the maximum-weight edge is minimized. A *bottleneck biconnected spanning subgraph* of  $G$  is a spanning subgraph,  $G'$ , of  $G$  which is biconnected and the weight of the longest edge in  $G'$  is minimized. For  $H \subseteq G$  we denote the bottleneck of  $H$ , i.e., the length of the longest edge in  $H$ , by  $\lambda(H)$ .

For a graph  $G = (V, E)$  and  $K \subseteq V$ , let  $G - K$  be the subgraph obtained from  $G$  by removing vertices in  $K$ , and let  $o(G - K)$  be the number of odd components in  $G - K$ . The following theorem by Tutte [16] gives a characterization of the graphs which have perfect matching:

**Theorem 1** (Tutte [16]).  *$G$  has a perfect matching if and only if  $o(G-K) \leq |K|$  for all  $K \subseteq V$ .*

Berge [5] extended Tutte's theorem to a formula (known as Tutte-Berge formula) for the maximum size of a matching in a graph. In a graph  $G$ , the *deficiency*,  $\text{def}_G(K)$ , is  $o(G-K) - |K|$ . Let  $\text{def}(G) = \max_{K \subseteq V} \text{def}_G(K)$ .

**Theorem 2** (Tutte-Berge formula; Berge [5]). *The size of a maximum matching in  $G$  is*

$$\frac{1}{2}(n - \text{def}(G)).$$

For an edge-weighted graph  $G$  we define the *weight sequence* of  $G$ ,  $\text{WS}(G)$ , as the sequence containing the weights of the edges of  $G$  in non-increasing order. A graph  $G_1$  is said to be less than a graph  $G_2$  if  $\text{WS}(G_1)$  is lexicographically smaller than  $\text{WS}(G_2)$ .

### 3 Connectivity

In this section we consider the connectivity of higher-order triangular-distance Delaunay graphs.

#### 3.1 $(k+1)$ -connectivity

For a set  $P$  of points in the plane, the TD-Delaunay graph, i.e., 0-TD, is not necessarily a triangulation [12], but it is connected and internally triangulated [4]. As shown in Figure 1(a), the outer face may not be convex and hence 0-TD is not necessarily biconnected. As a warm up exercise we show that every  $k$ -TD is  $(k+1)$ -connected.

**Theorem 3.** *For every point set  $P$ ,  $k$ -TD is  $(k+1)$ -connected. In addition, for every  $k$ , there exists a point set  $P$  such that  $k$ -TD is not  $(k+2)$ -connected.*

*Proof.* We prove the first part of this theorem by contradiction. Let  $K$  be the set of (at most)  $k$  vertices removed from  $k$ -TD, and let  $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ , where  $m > 1$ , be the resulting maximal connected components. Let  $T$  be the set of all triangles defined by any pair of points belonging to different components, i.e.,  $T = \{t(a, b) : a \in C_i, b \in C_j, i \neq j\}$ . Consider the smallest triangle  $t_{\min} \in T$ . Assume that  $t_{\min}$  is defined by two points  $a$  and  $b$ , i.e.,  $t_{\min} = t(a, b)$ , where  $a \in C_i$ ,  $b \in C_j$ , and  $i \neq j$ .

*Claim 1:*  $t_{\min}$  does not contain any point of  $P \setminus K$  in its interior. By contradiction, suppose that  $t_{\min}$  contains a point  $c \in P \setminus K$  in its interior. Three cases arise: (i)  $c \in C_i$ , (ii)  $c \in C_j$ , (iii)  $c \in C_l$ , where  $l \neq i$  and  $l \neq j$ . In case (i) the triangle  $t(c, b)$  between  $C_i$  and  $C_j$  is contained in  $t(a, b)$ . In case (ii) the triangle  $t(a, c)$  between  $C_i$  and  $C_j$  is contained in  $t(a, b)$ . In case (iii) both triangles  $t(a, c)$  and  $t(c, b)$  are contained in  $t(a, b)$ . All cases contradict the minimality of  $t(a, b) = t_{\min}$ . Thus,  $t_{\min}$  contains no point of  $P \setminus K$  in its interior, proving Claim 1.

By Claim 1,  $t_{\min} = t(a, b)$  may only contain points of  $K$ . Since  $|K| \leq k$ , there must be an edge between  $a$  and  $b$  in  $k$ -TD. This contradicts that  $a$  and  $b$  belong to different components  $C_i$  and  $C_j$  in  $\mathcal{C}$ . Therefore,  $k$ -TD is  $(k+1)$ -connected.

We present a constructive proof for the second part of theorem. Let  $P = A \cup B \cup K$ , where  $|A|, |B| \geq 1$  and  $|K| = k+1$ . Place the points of  $A$  in the plane. Let  $C_A^4 = \bigcap_{p \in A} C_p^4$ . Place the points of  $K$  in  $C_A^4$ . Let  $C_K^4 = \bigcap_{p \in K} C_p^4$ . Place the points of  $B$  in  $C_K^4$ . Consider any pair  $(a, b)$  of points where  $a \in A$  and  $b \in B$ . It is obvious that any path between  $a$  and  $b$  in  $k$ -TD goes through the vertices in  $K$ . Thus by removing the vertices in  $K$ ,  $a$  and  $b$  become disconnected. Therefore,  $k$ -TD of  $P$  is not  $(k+2)$ -connected.  $\square$

### 3.2 Bottleneck Biconnected Spanning Graph

As shown in Figure 1(a), 0-TD may not be biconnected. By Theorem 3, 1-TD is biconnected. In this section we show that a bottleneck biconnected spanning graph of  $P$  is contained in 1-TD.

**Theorem 4.** *For every point set  $P$ , 1-TD contains a bottleneck biconnected spanning graph of  $P$ .*

*Proof.* Let  $\mathcal{G}$  be the set of all biconnected spanning graphs with vertex set  $P$ . We define a total order on the elements of  $\mathcal{G}$  by their weight sequence. If two elements have the same weight sequence, we break the ties arbitrarily to get a total order. Let  $G^* = (P, E)$  be a graph in  $\mathcal{G}$  with minimal weight sequence. Clearly,  $G^*$  is a bottleneck biconnected spanning graph of  $P$ . We will show that all edges of  $G^*$  are in 1-TD. By contradiction suppose that some edges in  $E$  do not belong to 1-TD, and let  $e = (a, b)$  be the longest one (by the area of the triangle  $t(a, b)$ ). If the graph  $G^* - \{e\}$  is biconnected, then by removing  $e$ , we obtain a biconnected spanning graph  $G$  with  $WS(G) < WS(G^*)$ ; contradicting the minimality of  $G^*$ . Thus, there is a pair  $(p, q)$  of points such that any cycle between  $p$  and  $q$  in  $G^*$  goes through  $e$ . Since  $(a, b) \notin 1\text{-TD}$ ,  $t(a, b)$  contains at least two points of  $P$ , say  $x$  and  $y$ . Let  $G$  be the graph obtained from  $G^*$  by removing the edge  $(a, b)$  and adding the edges  $(a, x)$ ,  $(b, x)$ ,  $(a, y)$ ,  $(b, y)$ . We show that in  $G$  there is a cycle between  $p$  and  $q$  which does not go through  $e$ . Consider a cycle  $C$  in  $G^*$  between two points  $p$  and  $q$  (which goes through  $e$ ). If none of  $x$  and  $y$  belong to  $C$ , then  $(C - \{(a, b)\}) \cup \{(a, x), (b, x)\}$  is a cycle in  $G$  between  $p$  and  $q$ . If one of  $x$  or  $y$ , say  $x$ , belongs to  $C$ , then  $(C - \{(a, b)\}) \cup \{(a, y), (b, y)\}$  is a cycle in  $G$  between  $p$  and  $q$ . If both  $x$  and  $y$  belong to  $C$ , consider the partition of  $C$  into four parts: (a) edge  $(a, b)$ , (b) path  $\delta_{bx}$  between  $b$  and  $x$ , (c) path  $\delta_{xy}$  between  $x$  and  $y$ , and (d) path  $\delta_{ya}$  between  $y$  and  $a$ . There are four cases:

1. None of  $p$  and  $q$  are on  $\delta_{xy}$ . Then  $\delta_{bx} \cup \delta_{ya} \cup \{(a, x), (b, y)\}$  is a cycle in  $G$  between  $p$  and  $q$ .
2. Both  $p$  and  $q$  are on  $\delta_{xy}$ . Then  $\delta_{xy} \cup \{(a, x), (a, y)\}$  is a cycle in  $G$  between  $p$  and  $q$ .
3. One of  $p$  and  $q$  is on  $\delta_{xy}$  and the other one is on  $\delta_{bx}$ . Then  $\delta_{bx} \cup \delta_{xy} \cup \{(b, y)\}$  is a cycle in  $G$  between  $p$  and  $q$ .
4. One of  $p$  and  $q$  is on  $\delta_{xy}$  and the other one is on  $\delta_{ya}$ . Then  $\delta_{xy} \cup \delta_{ya} \cup \{(a, x)\}$  is a cycle in  $G$  between  $p$  and  $q$ .

Thus, between any pair of points in  $G$  there exists a cycle, and hence  $G$  is biconnected. Since  $x$  and  $y$  are inside  $t(a, b)$ , by Observation 2,  $\max\{t(a, x), t(a, y), t(b, x), t(b, y)\} < t(a, b)$ . Therefore,  $WS(G) < WS(G^*)$ ; contradicting the minimality of  $G^*$ .  $\square$

## 4 Hamiltonicity

In this section we show that 7-TD contains a bottleneck Hamiltonian cycle. In addition, we will show that for some point sets, 5-TD does not contain any bottleneck Hamiltonian cycle.

**Theorem 5.** *For every point set  $P$ , 7-TD contains a bottleneck Hamiltonian cycle.*

*Proof.* Let  $\mathcal{H}$  be the set of all Hamiltonian cycles through the points of  $P$ . Define a total order on the elements of  $\mathcal{H}$  by their weight sequence. If two elements have exactly the same weight sequence, break ties arbitrarily to get a total order. Let  $H^* = a_0, a_1, \dots, a_{n-1}$  be a cycle in  $\mathcal{H}$  with minimal weight sequence. It is obvious that  $H^*$  is a bottleneck Hamiltonian cycle of  $P$ .

We will show that all the edges of  $H^*$  are in 7-TD. Consider any edge  $e = (a_i, a_{i+1})$  in  $H^*$  and let  $t(a_i, a_{i+1})$  be the triangle corresponding to  $e$  (all index manipulations are modulo  $n$ ).

*Claim 1:* None of the edges of  $H^*$  can be completely inside  $t(a_i, a_{i+1})$ . Suppose there is an edge  $f = (a_j, a_{j+1})$  inside  $t(a_i, a_{i+1})$ . Let  $H$  be a cycle obtained from  $H^*$  by deleting  $e$  and  $f$ , and adding  $(a_i, a_j)$  and  $(a_{i+1}, a_{j+1})$ . By Observation 2,  $t(a_i, a_{i+1}) > \max\{t(a_i, a_j), t(a_{i+1}, a_{j+1})\}$ , and hence  $WS(H) < WS(H^*)$ . This contradicts the minimality of  $H^*$ .

Therefore, we may assume that no edge of  $H^*$  lies completely inside  $t(a_i, a_{i+1})$ . Suppose there are  $w$  points of  $P$  inside  $t(a_i, a_{i+1})$ . Let  $U = u_1, u_2, \dots, u_w$  represent these points indexed in the order we would encounter them on  $H^*$  starting from  $a_i$ . Let  $S = s_1, s_2, \dots, s_w$  and  $R = r_1, r_2, \dots, r_w$  represent the vertices where  $s_i$  is the vertex preceding  $u_i$  on the cycle and  $r_i$  is the vertex succeeding  $u_i$  on the cycle. Without loss of generality assume that  $a_i \in C_{a_{i+1}}^4$ , and  $t(a_i, a_{i+1})$  is anchored at  $a_{i+1}$ , as shown in Figure 4.

*Claim 2:* For each  $r_j \in R$ ,  $t(r_j, a_{i+1}) \geq \max\{t(a_i, a_{i+1}), t(u_j, r_j)\}$ . Suppose there is a point  $r_j \in R$  such that  $t(r_j, a_{i+1}) < \max\{t(a_i, a_{i+1}), t(u_j, r_j)\}$ . Construct a new cycle  $H$  by removing the edges  $(u_j, r_j)$ ,  $(a_i, a_{i+1})$  and adding the edges  $(a_{i+1}, r_j)$  and  $(a_i, u_j)$ . Since the two new edges have length strictly less than  $\max\{t(a_i, a_{i+1}), t(u_j, r_j)\}$ ,  $WS(H) < WS(H^*)$ ; which is a contradiction.

*Claim 3:* For each pair  $r_j$  and  $r_k$  of points in  $R$ ,  $t(r_j, r_k) \geq \max\{t(a_i, a_{i+1}), t(u_j, r_j), t(u_k, r_k)\}$ . Suppose there is a pair  $r_j$  and  $r_k$  such that  $t(r_j, r_k) < \max\{t(a_i, a_{i+1}), t(u_j, r_j), t(u_k, r_k)\}$ . Construct a new cycle  $H$  from  $H^*$  by first deleting  $(u_j, r_j)$ ,  $(u_k, r_k)$ ,  $(a_i, a_{i+1})$ . This results in three paths. One of the paths must contain both  $a_i$  and either  $r_j$  or  $r_k$ . W.l.o.g. suppose that  $a_i$  and  $r_k$  are on the same path. Add the edges  $(a_i, u_j)$ ,  $(a_{i+1}, u_k)$ ,  $(r_j, r_k)$ . Since  $\max\{t(u_j, r_j), t(u_k, r_k), d(a_i, a_{i+1})\} > \max\{t(a_i, u_j), t(a_{i+1}, u_k), t(r_j, r_k)\}$ ,  $WS(H) < WS(H^*)$ ; which is a contradiction.

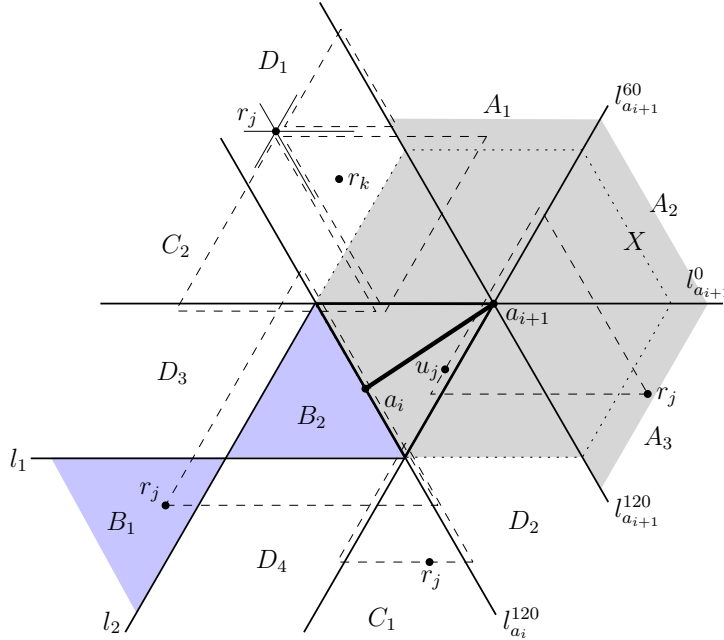


Figure 4: Illustration of Theorem 5.

Now, we use Claim 2 and Claim 3 to show that the size of  $R$  (and consequently  $U$ ) is at most seven, i.e.,  $w \leq 7$ . Consider the lines  $l_{a_{i+1}}^0$ ,  $l_{a_{i+1}}^{60}$ ,  $l_{a_{i+1}}^{120}$ , and  $l_{a_i}^{120}$  as shown in Figure 4. Let  $l_1$  and  $l_2$  be the rays starting at the corners of  $t(a_i, a_{i+1})$  opposite to  $a_{i+1}$  and parallel to  $l_{a_{i+1}}^0$  and  $l_{a_{i+1}}^{60}$  respectively, as shown in Figure 4. These lines and rays, partition the plane into 12 regions. We

will show that each of the regions  $D_1, D_2, D_3, D_4, C_1, C_2$ , and  $B = B_1 \cup B_2$  contains at most one point of  $R$ , and the other regions do not contain any point of  $R$ . Consider the hexagon  $X(a_{i+1}, a_i)$ . By Claim 2 and Observation 4, no point of  $R$  can be inside  $X(a_{i+1}, a_i)$ . Moreover, no point of  $R$  can be inside the cones  $A_1, A_2$ , and  $A_3$ , because if  $r_j \in \{A_1 \cup A_2 \cup A_3\}$ , the (upward) triangle  $t'(u_j, r_j)$  contains  $a_{i+1}$ . Then by Observation 4,  $t(r_j, a_{i+1}) < t(u_j, r_j)$ ; which contradicts Claim 2.

Now we show that each of the regions  $D_1, D_2, D_3$  and  $D_4$  contains at most one point of  $R$ . Consider the region  $D_1$ ; by similar reasoning we can prove this claim for  $D_2, D_3$ , and  $D_4$ . Using contradiction, let  $r_j$  and  $r_k$  be two points in  $D_1$ , and w.l.o.g. assume that  $r_j$  is the farthest to  $l_{a_{i+1}}^{60}$ . Then  $r_k$  can lie inside any of the cones  $C_{r_j}^1, C_{r_j}^5$ , and  $C_{r_j}^6$  (but not in  $X$ ). If  $r_k \in C_{r_j}^1$ , then  $t'(r_j, r_k)$  is smaller than  $t'(a_i, a_{i+1})$  which means that  $t(r_j, r_k) < t(a_i, a_{i+1})$ . If  $r_k \in C_{r_j}^5$ , then  $t'(u_j, r_j)$  contains  $r_k$ , that is  $t(r_j, r_k) < t(u_j, r_j)$ . If  $r_k \in C_{r_j}^6$ , then  $t(u_j, r_j)$  contains  $r_k$ , that is  $t(r_j, r_k) < t(u_j, r_j)$ . All cases contradict Claim 3.

Now consider the region  $C_1$  (or its symmetric region  $C_2$ ) and by contradiction assume that it contains two points  $r_j$  and  $r_k$ . Let  $r_j$  be the farthest from  $l_{a_{i+1}}^0$ . It is obvious that the  $t'(u_j, r_j)$  contains  $r_k$ , that is  $t(r_j, r_k) < t(u_j, r_j)$ ; which contradicts Claim 3.

Now consider the region  $B = B_1 \cup B_2$ . If both  $r_j$  and  $r_k$  belong to  $B_2$ , then  $t'(r_j, r_k)$  is smaller than  $t(a_i, a_{i+1})$ . If  $r_j \in B_1$  and  $r_k \in B_2$ , then  $t'(u_j, r_j)$  contains  $r_k$ , and hence  $t(r_j, r_k) < t(u_j, r_j)$ . If both  $r_j$  and  $r_k$  belong to  $B_1$ , let  $r_j$  be the farthest from  $l_{a_i}^{120}$ . Clearly,  $t(u_j, r_j)$  contains  $r_k$  and hence  $t(r_j, r_k) < t(u_j, r_j)$ . All cases contradict Claim 3.

Therefore, any of the regions  $D_1, D_2, D_3, D_4, C_1, C_2$ , and  $B = B_1 \cup B_2$  contains at most one point of  $R$ . Thus,  $w \leq 7$ , and  $t(a_i, a_{i+1})$  contains at most 7 points of  $P$ . Therefore,  $e = (a_i, a_{i+1})$  is an edge of 7-TD.  $\square$

As a direct consequence of Theorem 5 we have shown that:

**Corollary 1.** *7-TD is Hamiltonian.*

An interesting question is to determine if  $k$ -TD contains a bottleneck Hamiltonian cycle for  $k < 7$ . Figure 5 shows a configuration where  $t(a_i, a_{i+1})$  contains 7 points while the conditions of Claim 1, Claim 2, and Claim 3 in the proof of Theorem 5 hold. In Figure 5,  $d(a_i, a_{i+1}) = 1$ ,  $d(r_i, u_i) = 1 + \epsilon$ ,  $d(r_i, r_j) > 1 + \epsilon$ ,  $d(r_i, a_{i+1}) > 1 + \epsilon$  for  $i, j = 1, \dots, 7$  and  $i \neq j$ .

Figure 6 shows a configuration of  $P$  with 17 points such that 5-TD does not contain a bottleneck Hamiltonian cycle. In Figure 6,  $d(a, b) = 1$  and  $t(a, b)$  contains 6 points  $U = \{u_1, \dots, u_6\}$ . In addition  $d(r_i, u_i) = 1 + \epsilon$ ,  $d(r_i, r_j) > 1 + \epsilon$ ,  $d(r_i, b) > 1 + \epsilon$  for  $i, j = 1, \dots, 6$  and  $i \neq j$ . Let  $R = \{t_1, t_2, t_3, r_1, \dots, r_6\}$ . The dashed hexagons are centered at  $a$  and  $b$  and have diameter 1. The dotted hexagons are centered at vertices in  $R$  and have diameter  $1 + \epsilon$ . Each point in  $R$  is connected to its first and second closest points by edges of length  $1 + \epsilon$  (the bold edges). Let  $B$  be the set of these edges. Let  $H$  be a cycle formed by  $B \cup \{(u_3, b), (b, a), (a, u_5)\}$ , i.e.,  $H = (u_4, r_4, u_5, r_5, u_6, r_6, t_1, t_2, t_3, r_1, u_1, r_2, u_2, r_3, u_3, a, b, u_4)$ . It is obvious that  $H$  is a Hamiltonian cycle for  $P$  and  $\lambda(H) = 1 + \epsilon$ . Thus, the bottleneck of any bottleneck Hamiltonian cycle for  $P$  is at most  $1 + \epsilon$ . We will show that any bottleneck Hamiltonian cycle for  $P$  contains the edge  $(a, b)$  which does not belong to 5-TD. By contradiction, let  $H^*$  be a bottleneck Hamiltonian cycle which does not contain  $(a, b)$ . In  $H^*$ ,  $b$  is connected to two vertices  $b_l$  and  $b_r$ , where  $b_l \neq a$  and  $b_r \neq a$ . Since the distance between  $b$  and any vertex in  $R$  is strictly bigger than  $1 + \epsilon$  and  $\lambda(H^*) \leq 1 + \epsilon$ ,  $b_l \notin R$  and  $b_r \notin R$ . Thus  $b_l$  and  $b_r$  belong to  $U$ . Let  $U' = \{u_1, u_2, u_5, u_6\}$ . Consider two cases:

- $b_l \in U'$  or  $b_r \in U'$ . W.l.o.g. assume that  $b_l \in U'$  and  $b_l = u_1$ . Since  $u_1$  is the first/second closest point of  $r_1$  and  $r_2$ , in  $H^*$  one of  $r_1$  and  $r_2$  must be connected by an edge  $e$  to a point that is farther than its second closet point;  $e$  has length strictly greater than  $1 + \epsilon$ .



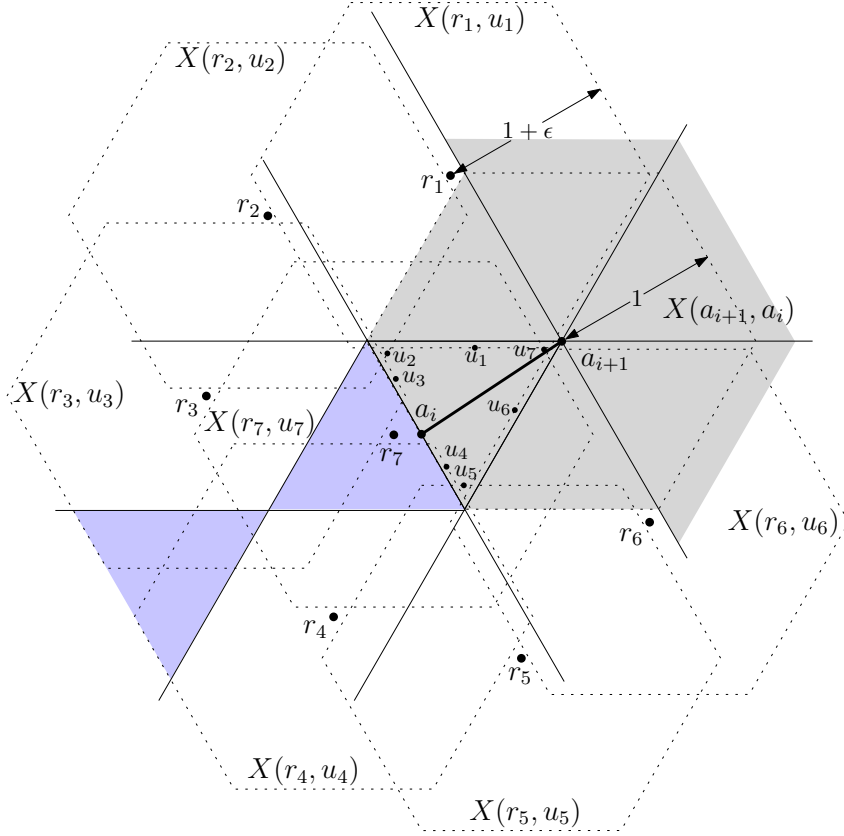


Figure 5:  $t(a_i, a_{i+1})$  contains 7 points while the conditions in the proof of Theorem 5 hold.

- $b_l \notin U'$  and  $b_r \notin U'$ . Thus, both  $b_l$  and  $b_r$  belong to  $\{u_3, u_4\}$ . That is, in  $H^*$ ,  $a$  should be connected to a point  $c$  where  $c \in R \cup U'$ . If  $c \in R$  then the edge  $(a, c)$  has length more than  $1 + \epsilon$ . If  $c \in U'$ , w.l.o.g. assume  $c = u_1$ ; by the same argument as in the previous case, one of  $r_1$  and  $r_2$  must be connected by an edge  $e$  to a point that is farther than its second closet point;  $e$  has length strictly greater than  $1 + \epsilon$ .

Since  $e \in H^*$ , both cases contradicts that  $\lambda(H^*) \leq 1 + \epsilon$ . Therefore, every bottleneck Hamiltonian cycle contains edge  $(a, b)$ . Since  $(a, b)$  is not an edge in 5-TD, a bottleneck Hamiltonian cycle of  $P$  is not contained in 5-TD.

## 5 Perfect Matching Admissibility

In this section we consider the matching problem in higher-order triangular-distance Delaunay graphs. In Subsection 5.1 we show that 6-TD contains a bottleneck perfect matching. We also show that for some point sets  $P$ , 5-TD does not contain any bottleneck perfect matching. In Subsection 5.2 we prove that every 2-TD has a perfect matching when  $P$  has an even number of points, and 1-TD contains a matching of size at least  $\frac{2(n-1)}{5}$ .

### 5.1 Bottleneck Perfect Matching

**Theorem 6.** *For a set  $P$  of an even number of points, 6-TD contains a bottleneck perfect matching.*

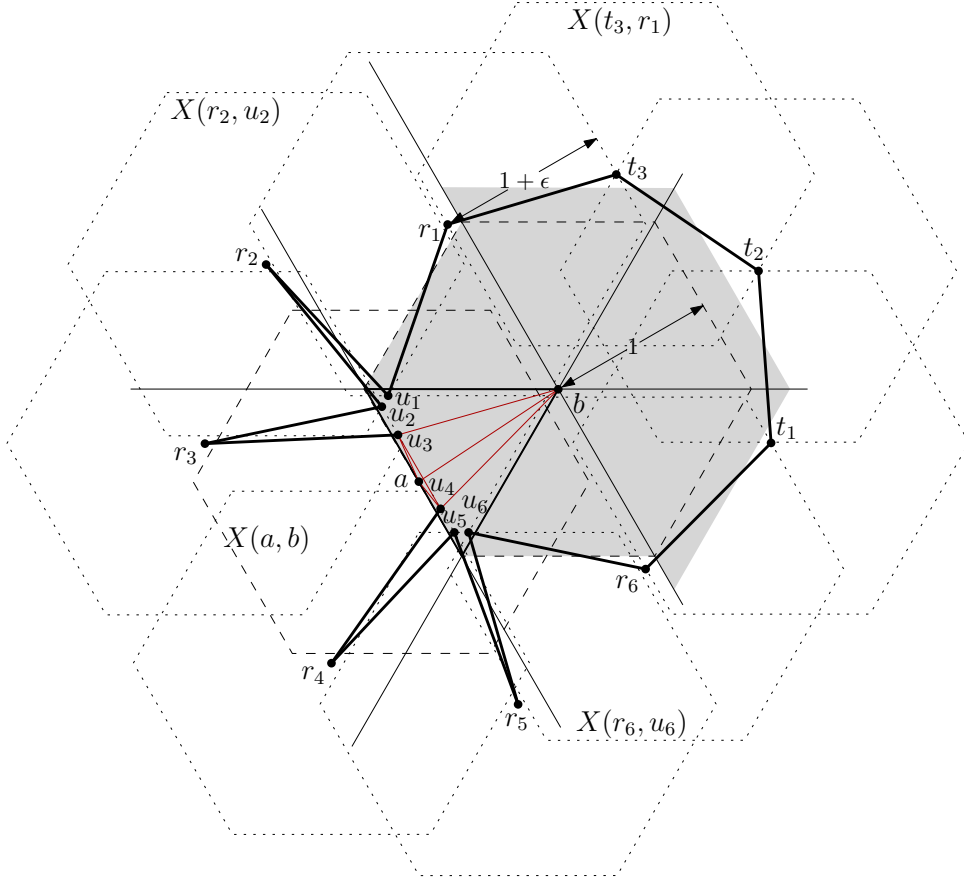


Figure 6: The points  $\{r_1, \dots, r_6, t_1, t_2, t_3\}$  are connected to their first and second closest point (the bold edges). The edge  $(a, b)$  should be in any bottleneck Hamiltonian cycle, while  $t(a, b)$  contains 6 points.

*Proof.* Let  $\mathcal{M}$  be the set of all perfect matchings through the points of  $P$ . Define a total order on the elements of  $\mathcal{M}$  by their weight sequence. If two elements have exactly the same weight sequence, break ties arbitrarily to get a total order. Let  $M^* = \{(a_1, b_1), \dots, (a_{\frac{n}{2}}, b_{\frac{n}{2}})\}$  be a perfect matching in  $\mathcal{M}$  with minimal weight sequence. It is obvious that  $M^*$  is a bottleneck perfect matching for  $P$ . We will show that all edges of  $M^*$  are in 6-TD. Consider any edge  $e = (a_i, b_i)$  in  $M^*$  and its corresponding triangle  $t(a_i, b_i)$ .

*Claim 1:* None of the edges of  $M^*$  can be inside  $t(a_i, b_i)$ . Suppose there is an edge  $f = (a_j, b_j)$  inside  $t(a_i, b_i)$ . Let  $M$  be a perfect matching obtained from  $M^*$  by deleting  $\{e, f\}$ , and adding  $\{(a_i, a_j), (b_i, b_j)\}$ . By Observation 2, the two new edges are smaller than the old ones. Thus,  $WS(M) < WS(M^*)$  which contradicts the minimality of  $M^*$ .

Therefore, we may assume that no edge of  $M^*$  lies completely inside  $t(a_i, b_i)$ . Suppose there are  $w$  points of  $P$  inside  $t(a_i, b_i)$ . Let  $U = u_1, u_2, \dots, u_w$  represent the points inside  $t(a_i, b_i)$ , and  $R = r_1, r_2, \dots, r_w$  represent the points where  $(r_i, u_i) \in M^*$ . W.l.o.g. assume that  $a_i \in C_{b_i}^4$ , and  $t(a_i, b_i)$  is anchored at  $b_i$  as shown in Figure 7.

*Claim 2:* For each  $r_j \in R$ ,  $\min\{t(r_j, a_i), t(r_j, b_i)\} \geq \max\{t(a_i, b_i), t(u_j, r_j)\}$ . By a similar argument as in the proof of Claim 2 in Theorem 5 we can either match  $r_j$  with  $a_i$  or  $b_i$  to obtain a smaller matching  $M$ ; which is a contradiction.

*Claim 3:* For each pair  $r_j$  and  $r_k$  of points in  $R$ ,  $t(r_j, r_k) \geq \max\{t(a_i, b_i), t(r_j, u_j), t(r_k, u_k)\}$ . The proof is similar to the proof of Claim 3 in Theorem 5.

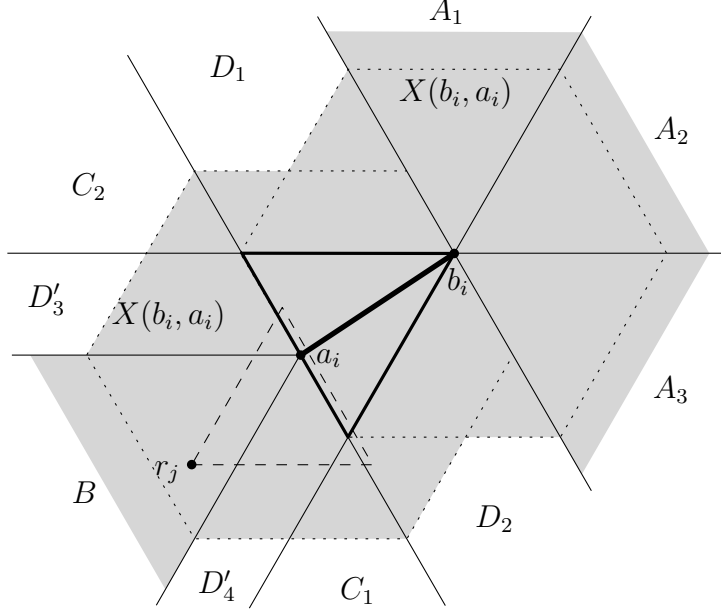


Figure 7: Proof of Theorem 6.

Consider Figure 7 which partitions the plane into eleven regions. As a direct consequence of Claim 2, the hexagons  $X(b_i, a_i)$  and  $X(a_i, b_i)$  do not contain any point of  $R$ . By a similar argument as in the proof of Theorem 5, the regions  $A_1, A_2, A_3$  do not contain any point of  $R$ . In addition, the region  $B$  does not contain any point  $r_j$  of  $R$ , because otherwise  $t'(r_j, u_j)$  contains  $a_i$ , that is  $t(r_j, a_i) < t(u_j, r_j)$  which contradicts Claim 2. As shown in the proof of Theorem 5 each of the regions  $D_1, D_2, D'_3, D'_4, C_1$ , and  $C_2$  contains at most one point of  $R$  (note that  $D'_3 \subset D_3$  and  $D'_4 \subset D_4$ ). Thus,  $w \leq 6$ , and  $t(a_i, b_i)$  contains at most 6 points of  $P$ . Therefore,  $e = (a_i, b_i)$  is an edge of 6-TD.  $\square$

As a direct consequence of Theorem 6 we have shown that:

**Corollary 2.** *For a set  $P$  of even number of points, 6-TD has a perfect matching.*

We show that the bound  $k = 6$  proved in Theorem 6 is tight. We will show that there are point sets  $P$  such that 5-TD does not contain any bottleneck perfect matching. Figure 8 shows a configuration of  $P$  with 14 points such that  $d(a, b) = 1$  and  $t(a, b)$  contains six points  $U = \{u_1, \dots, u_6\}$ . In addition  $d(r_i, u_i) = 1 + \epsilon$ ,  $d(r_i, x) > 1 + \epsilon$  where  $x \neq u_i$ , for  $i = 1, \dots, 6$ . Let  $R = \{r_1, \dots, r_6\}$ . In Figure 8, the dashed hexagons are centered at  $a$  and  $b$ , each of diameter 1, and the dotted hexagons centered at vertices in  $R$ , each of diameter  $1 + \epsilon$ . Consider a perfect matching  $M = \{(a, b)\} \cup \{(r_i, u_i) : i = 1, \dots, 6\}$  where each point  $r_i \in R$  is matched to its closest point  $u_i$ . It is obvious that  $\lambda(M) = 1 + \epsilon$ , and hence the bottleneck of any bottleneck perfect matching is at most  $1 + \epsilon$ . We will show that any bottleneck perfect matching for  $P$  contains the edge  $(a, b)$  which does not belong to 5-TD. By contradiction, let  $M^*$  be a bottleneck perfect matching which does not contain  $(a, b)$ . In  $M^*$ ,  $b$  is matched to a point  $c \in R \cup U$ . If  $c \in R$ , then  $d(b, c) > 1 + \epsilon$ . If  $c \in U$ , w.l.o.g. assume  $c = u_1$ . Thus, in  $M^*$  the point  $r_1$  is matched to a point  $d$  where  $d \neq u_1$ . Since  $u_1$  is the closest point to  $r_1$  and  $d(r_1, u_1) = 1 + \epsilon$ ,  $d(r_1, d) > 1 + \epsilon$ . Both cases contradicts that  $\lambda(M^*) \leq 1 + \epsilon$ . Therefore, every bottleneck perfect matching contains  $(a, b)$ . Since  $(a, b)$  is not an edge in 5-TD, a bottleneck perfect matching of  $P$  is not contained in 5-TD.

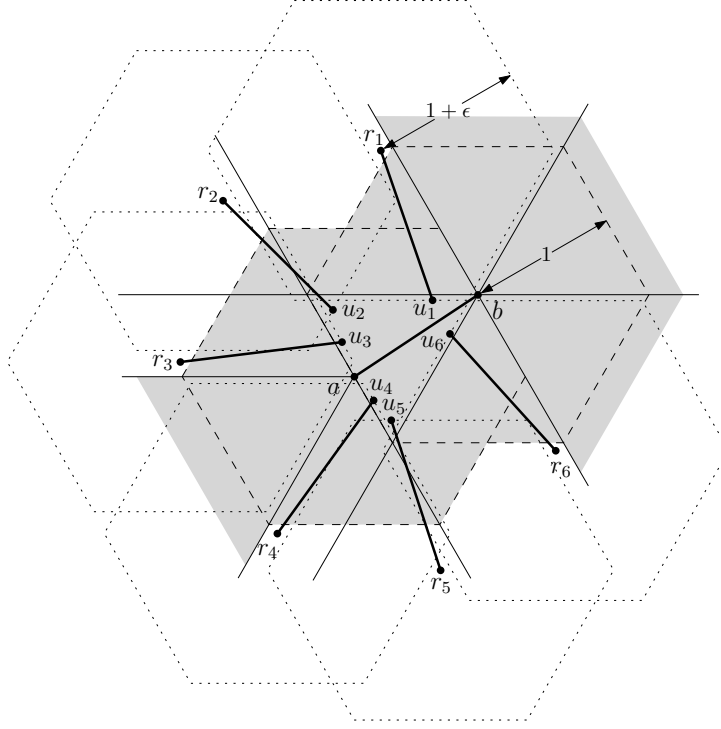


Figure 8: The points  $\{r_1, \dots, r_6\}$  are matched to their closest point. The edge  $(a, b)$  should be an edge in any bottleneck perfect matching, while  $t(a, b)$  contains 6 points.

## 5.2 Perfect Matching

In [4] the authors proved a tight lower bound of  $\lceil \frac{n-1}{3} \rceil$  on the size of a maximum matching in 0-TD. In this section we prove that 1-TD has a matching of size  $\frac{2(n-1)}{5}$  and 2-TD has a perfect matching when  $P$  has an even number of points.

For a triangle  $t(a, b)$  through the points  $a$  and  $b$ , let  $top(a, b)$ ,  $left(a, b)$ , and  $right(a, b)$  respectively denote the top, left, and right sides of  $t(a, b)$ . Refer to Figure 9(a) for the following lemma.

**Lemma 1.** *Let  $t(a, b)$  and  $t(p, q)$  intersect a horizontal line  $\ell$ , and  $t(a, b)$  intersects  $top(p, q)$  in such a way that  $t(p, q)$  contains the lowest corner of  $t(a, b)$ . If  $a$  and  $b$  lie above  $top(p, q)$ , and  $p$  and  $q$  lie above  $\ell$ , then,  $\max\{t(a, p), t(b, q)\} < \max\{t(a, b), t(p, q)\}$ .*

*Proof.* Recall that  $t(a, b)$  is the smallest downward triangle through  $a$  and  $b$ . By Observation 1 each side of  $t(a, b)$  contains either  $a$  or  $b$ . In Figure 9(a) the set of potential positions for point  $a$  on the boundary of  $t(a, b)$  is shown by the line segment  $s_a$ ; and similarly by  $s_b$ ,  $s_p$ ,  $s_q$  for  $b$ ,  $p$ ,  $q$ , respectively. We will show that  $t(a, p) < \max\{t(a, b), t(p, q)\}$ . By similar reasoning we can show that  $t(b, q) < \max\{t(a, b), t(p, q)\}$ . Let  $x$  denote the intersection of  $\ell$  and  $right(p, q)$ . Consider a ray  $r$  initiated at  $x$  and parallel to  $left(p, q)$  which divides  $s_a$  into (at most) two parts  $s'_a$  and  $s''_a$  as shown in Figure 9(b). Two cases may appear:

- $a \in s'_a$ . Let  $t_1$  be a downward triangle anchored at  $x$  which has its  $top$  side on the line through  $top(a, b)$  (the dashed triangle in Figure 9(b)). The top side of  $t_1$  and  $t(a, b)$  lie on the same horizontal line. The bottommost corner of  $t_1$  is on  $\ell$  while the bottommost corner of  $t(a, b)$  is below  $\ell$ . Thus,  $t_1 < t(a, b)$ . In addition,  $t_1$  contains  $s'_a$  and  $s_p$ , thus, for any two points  $a \in s'_a$  and  $p \in s_p$ ,  $t(a, p) \leq t_1$ . Therefore,  $t(a, p) < t(a, b)$ .

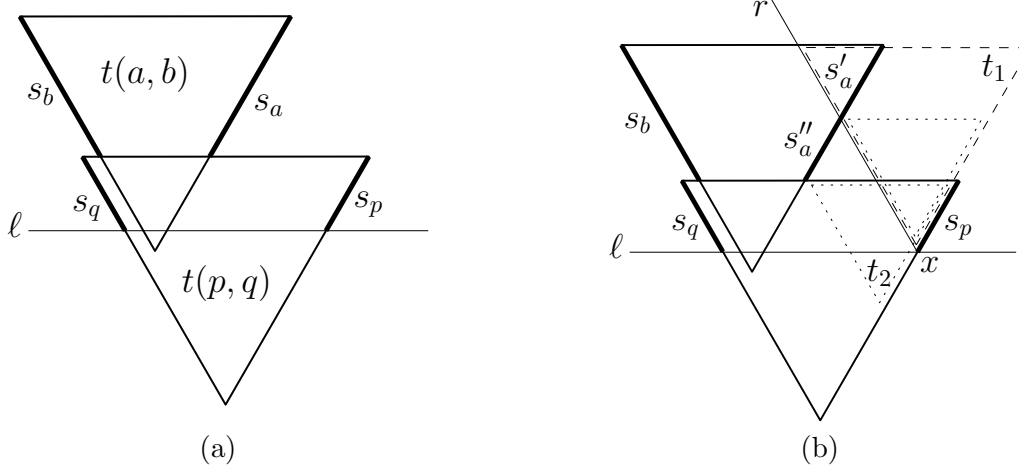


Figure 9: (a) Illustration of Lemma 1, and (b) proof of Lemma 1.

- $a \in s''_a$ . Let  $t_2$  be a downward triangle anchored at the intersection of  $right(a, b)$  and  $top(p, q)$  which has one side on the line through  $right(p, q)$  (the dotted triangle in Figure 9(b)). This triangle is contained in  $t(p, q)$ , and has  $s_p$  on its right side. If we slide  $t_2$  upward while its top-left corner remains on  $s''_a$ , the segment  $s_p$  remains on the right side of  $t_2$ . Thus, any triangle connecting a point  $a \in s''_a$  to a point  $p \in s_p$  has the same size as  $t_2$ . That is,  $t(a, p) = t_2 < t(p, q)$ .

Therefore, we have  $t(a, p) < \max\{t(a, b), t(p, q)\}$ . By similar argument we conclude that  $t(b, q) < \max\{t(a, b), t(p, q)\}$ .  $\square$

Let  $\mathcal{P} = \{P_1, P_2, \dots\}$  be a partition of the points in  $P$ . Let  $G(\mathcal{P})$  be a complete graph with vertex set  $\mathcal{P}$ . For each edge  $e = (P_i, P_j)$  in  $G(\mathcal{P})$ , let  $w(e)$  be equal to the area of the smallest triangle between a point in  $P_i$  and a point in  $P_j$ , i.e.  $w(e) = \min\{t(a, b) : a \in P_i, b \in P_j\}$ . That is, the weight of an edge  $e \in G(\mathcal{P})$  corresponds to the size of the smallest triangle  $t(e)$  defined by the endpoints of  $e$ . Let  $\mathcal{T}$  be a minimum spanning tree of  $G(\mathcal{P})$ . Let  $T$  be the set of triangles corresponding to the edges of  $\mathcal{T}$ , i.e.  $T = \{t(e) : e \in \mathcal{T}\}$ .

**Lemma 2.** *The interior of any triangle in  $T$  does not contain any point of  $P$ .*

*Proof.* By contradiction, suppose there is a triangle  $\tau \in T$  which contains a point  $c \in P$ . Let  $e = (P_i, P_j)$  be the edge in  $\mathcal{T}$  which corresponds to  $\tau$ . Let  $a$  and  $b$  respectively be the points in  $P_i$  and  $P_j$  which define  $\tau$ , i.e.  $\tau = t(a, b)$  and  $w(e) = t(a, b)$ . Three cases arise: (i)  $c \in P_i$ , (ii)  $c \in P_j$ , (iii)  $c \in P_l$  where  $l \neq i$  and  $l \neq j$ . In case (i) the triangle  $t(c, b)$  between  $c \in P_i$  and  $b \in P_j$  is smaller than  $t(a, b)$ ; contradicts that  $w(e) = t(a, b)$  in  $G(\mathcal{P})$ . In case (ii) the triangle  $t(a, c)$  between  $a \in P_i$  and  $c \in P_j$  is smaller than  $t(a, b)$ ; contradicts that  $w(e) = t(a, b)$  in  $G(\mathcal{P})$ . In case (iii) the triangle  $t(a, c)$  (resp.  $t(c, b)$ ) between  $P_i$  and  $P_l$  (resp.  $P_l$  and  $P_j$ ) is smaller than  $t(a, b)$ ; contradicts that  $e$  is an edge in  $\mathcal{T}$ .  $\square$

**Lemma 3.** *Each point in the plane can be in the interior of at most three triangles in  $T$ .*

*Proof.* For each  $t(a, b) \in T$ , the sides  $top(a, b)$ ,  $right(a, b)$ , and  $left(a, b)$  contains at least one of  $a$  and  $b$ . In addition, by Lemma 2,  $t(a, b)$  does not contain any point of  $P$  in its interior. Thus, none of  $top(a, b)$ ,  $right(a, b)$ , and  $left(a, b)$  is completely inside the other triangles. Therefore, the only possible way that two triangles  $t(a, b)$  and  $t(p, q)$  can share a point is that one triangle, say  $t(p, q)$ , contains a corner of  $t(a, b)$  in such a way that  $a$  and  $b$  are outside  $t(p, q)$ . In other

words  $t(a, b)$  intersects  $t(p, q)$  through one of the sides  $top(p, q)$ ,  $right(p, q)$ , or  $left(p, q)$ . If  $t(a, b)$  intersects  $t(p, q)$  through a direction  $d \in \{top, right, left\}$  we say that  $t(p, q) \prec_d t(a, b)$ .

By contradiction, suppose there is a point  $c$  in the plane which is inside four triangles  $\{t_1, t_2, t_3, t_4\} \subseteq T$ . Out of these four, either (i) three of them are like  $t_i \prec_d t_j \prec_d t_k$  or (ii) there is a triangle  $t_l$  such that  $t_l \prec_{top} t_i, t_l \prec_{right} t_j, t_l \prec_{left} t_k$ , where  $1 \leq i, j, k, l \leq 4$  and  $i \neq j \neq k \neq l$ . Figure 10 shows the two possible configurations (note that all other configurations obtained by changing the indices of triangles and/or the direction are symmetric to Figure 10(a) or Figure 10(b)).

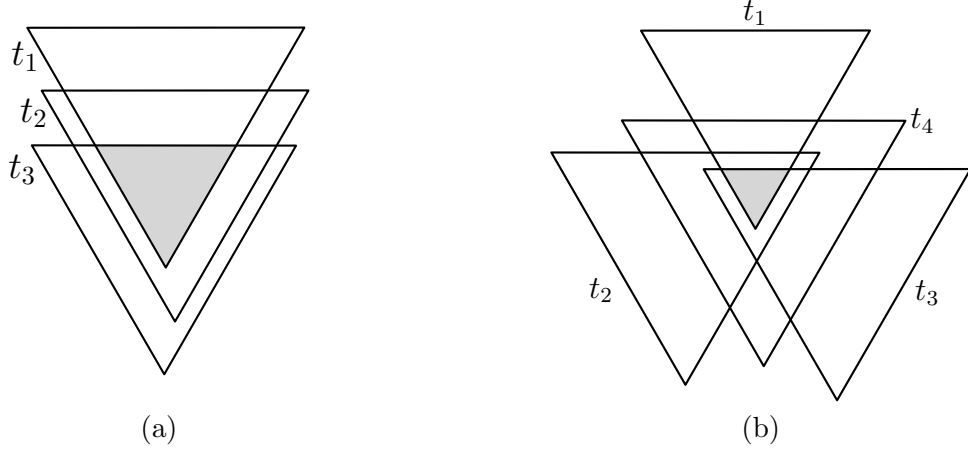


Figure 10: Two possible configurations: (a)  $t_3 \prec_{top} t_2 \prec_{top} t_1$ , (b)  $t_4 \prec_{top} t_1, t_4 \prec_{left} t_2, t_4 \prec_{right} t_3$ .

Recall that each of  $t_1, t_2, t_3, t_4$  corresponds to an edge in  $\mathcal{T}$ . In the configuration of Figure 10(a) consider  $t_1, t_2$ , and  $top(t_3)$  which is shown in more detail in Figure 11(a). Suppose  $t_1$  (resp.  $t_2$ ) is defined by points  $a$  and  $b$  (resp.  $p$  and  $q$ ). By Lemma 2,  $p$  and  $q$  are above  $top(t_3)$ ,  $a$  and  $b$  are above  $top(t_2)$ . By Lemma 1,  $\max\{t(a, p), t(b, q)\} < \max\{t(a, b), t(p, q)\}$ . This contradicts the fact that both of the edges representing  $t(a, b)$  and  $t(p, q)$  are in  $\mathcal{T}$ , because by replacing  $\max\{t(a, b), t(p, q)\}$  with  $t(a, p)$  or  $t(b, q)$ , we obtain a tree  $\mathcal{T}'$  which is smaller than  $\mathcal{T}$ . In the configuration of Figure 10(b), consider all pairs of potential positions for two points defining  $t_4$  which is shown in more detail in Figure 11(b). The pairs of potential positions on the boundary of  $t_4$  are shown in red, green, and orange. Consider the red pair, and look at  $t_2, t_4$ , and  $left(t_1)$ . By Lemma 1 and the same reasoning as for the previous configuration, we obtain a smaller tree  $\mathcal{T}'$ ; which contradicts the minimality of  $\mathcal{T}$ . By symmetry, the green and orange pairs lead to a contradiction. Therefore, all configurations are invalid; which proves the lemma.  $\square$

Our results in this section are based on Lemma 2, Lemma 3 and the two theorems by Tutte [16] and Berge [5].

Now we prove that 2-TD has a perfect matching.

**Theorem 7.** *For a set  $P$  of an even number of points, 2-TD has a perfect matching.*

*Proof.* First we show that by removing a set  $K$  of  $k$  points from 2-TD, at most  $k+1$  components are generated. Then we show that at least one of these components must be even. Finally by Theorem 1 we conclude that 2-TD has a perfect matching.

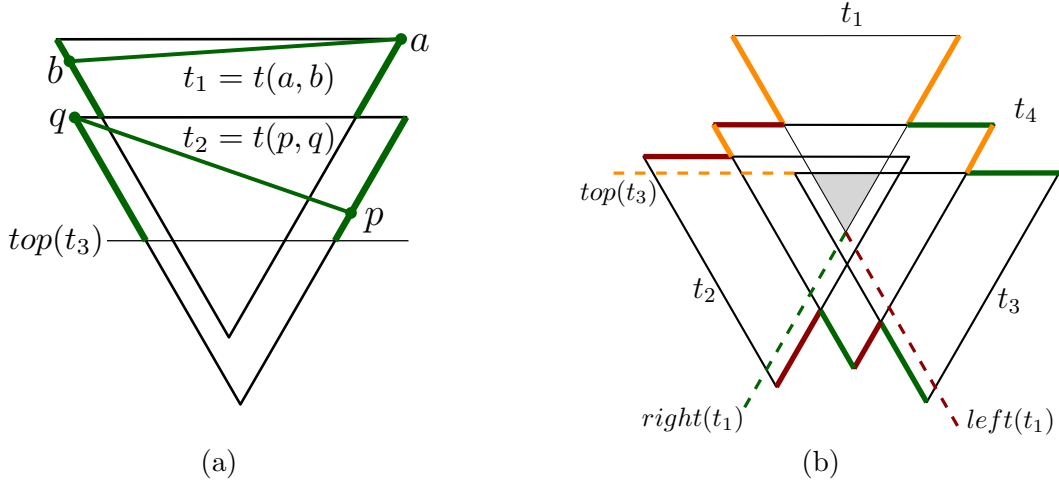


Figure 11: Illustration of Lemma 3.

Let  $K$  be a set of  $k$  vertices removed from 2-TD, and let  $\mathcal{C} = \{C_1, \dots, C_{m(k)}\}$  be the resulting  $m(k)$  components, where  $m$  is a function depending on  $k$ . Actually  $\mathcal{C} = 2\text{-TD} - K$  and  $\mathcal{P} = \{V(C_1), \dots, V(C_{m(k)})\}$  is a partition of the vertices in  $P \setminus K$ .

**Claim 1.**  $m(k) \leq k + 1$ . Let  $G(\mathcal{P})$  be a complete graph with vertex set  $\mathcal{P}$  which is constructed as described above. Let  $\mathcal{T}$  be a minimum spanning tree of  $G(\mathcal{P})$  and let  $T$  be the set of triangles corresponding to the edges of  $\mathcal{T}$ . It is obvious that  $\mathcal{T}$  contains  $m(k) - 1$  edges and hence  $|T| = m(k) - 1$ . Let  $F = \{(p, t) : p \in K, t \in T, p \in t\}$  be the set of all (point, triangle) pairs where  $p \in K$ ,  $t \in T$ , and  $p$  is inside  $t$ . By Lemma 3 each point in  $K$  can be inside at most three triangles in  $T$ . Thus,  $|F| \leq 3 \cdot |K|$ . Now we show that each triangle in  $T$  contains at least three points of  $K$ . Consider any triangle  $\tau \in T$ . Let  $e = (V(C_i), V(C_j))$  be the edge of  $\mathcal{T}$  which is corresponding to  $\tau$ , and let  $a \in V(C_i)$  and  $b \in V(C_j)$  be the points defining  $\tau$ . By Lemma 2,  $\tau$  does not contain any point of  $P \setminus K$  in its interior. Therefore,  $\tau$  contains at least three points of  $K$ , because otherwise  $(a, b)$  is an edge in 2-TD which contradicts the fact that  $a$  and  $b$  belong to different components in  $\mathcal{C}$ . Thus, each triangle in  $T$  contains at least three points of  $K$  in its interior. That is,  $3 \cdot |T| \leq |F|$ . Therefore,  $3(m(k) - 1) \leq |F| \leq 3k$ , and hence  $m(k) \leq k + 1$ .

**Claim 2:**  $o(\mathcal{C}) \leq k$ . By Claim 1,  $|\mathcal{C}| = m(k) \leq k + 1$ . If  $|\mathcal{C}| \leq k$ , then  $o(\mathcal{C}) \leq k$ . Assume that  $|\mathcal{C}| = k + 1$ . Since  $P = K \cup \{\bigcup_{i=1}^{k+1} V(C_i)\}$ , the total number of vertices of  $P$  can be defined as  $n = k + \sum_{i=1}^{k+1} |V(C_i)|$ . Consider two cases where (i)  $k$  is odd, (ii)  $k$  is even. In both cases if all the components in  $\mathcal{C}$  are odd, then  $n$  is odd; contradicts our assumption that  $P$  has an even number of vertices. Thus,  $\mathcal{C}$  contains at least one even component, which implies that  $o(\mathcal{C}) \leq k$ .

Finally, by Claim 2 and Theorem 1, we conclude that 2-TD has a perfect matching.  $\square$

**Theorem 8.** For every set  $P$  of points, 1-TD has a matching of size  $\frac{2(n-1)}{5}$ .

*Proof.* Let  $K$  be a set of  $k$  vertices removed from 1-TD, and let  $\mathcal{C} = \{C_1, \dots, C_{m(k)}\}$  be the resulting  $m(k)$  components. Actually  $\mathcal{C} = 1\text{-TD} - K$  and  $\mathcal{P} = \{V(C_1), \dots, V(C_{m(k)})\}$  is a partition of the vertices in  $P \setminus K$ . Note that  $o(\mathcal{C}) \leq m(k)$ . Let  $M^*$  be a maximum matching in 1-TD. By Theorem 2,

$$|M^*| = \frac{1}{2}(n - \text{def}(1\text{-TD})), \quad (1)$$

where

$$\begin{aligned}
\text{def}(1\text{-TD}) &= \max_{K \subseteq P} (o(\mathcal{C}) - |K|) \\
&\leq \max_{K \subseteq P} (|\mathcal{C}| - |K|) \\
&= \max_{0 \leq k \leq n} (m(k) - k).
\end{aligned} \tag{2}$$

Define  $G(\mathcal{P})$ ,  $\mathcal{T}$ ,  $T$ , and  $F$  as in the proof of Theorem 7. By Lemma 3,  $|F| \leq 3 \cdot |K|$ . By the same reasoning as in the proof of Theorem 7, each triangle in  $T$  has at least two points of  $K$  in its interior. Thus,  $2 \cdot |T| \leq |F|$ . Therefore,  $2(m(k) - 1) \leq |F| \leq 3k$ , and hence

$$m(k) \leq \frac{3k}{2} + 1. \tag{3}$$

In addition,  $k + m(k) = |K| + |\mathcal{C}| \leq |P| = n$ , and hence

$$m(k) \leq n - k. \tag{4}$$

By Inequalities (3) and (4),

$$m(k) \leq \min\left\{\frac{3k}{2} + 1, n - k\right\}. \tag{5}$$

Thus, by (2) and (5)

$$\begin{aligned}
\text{def}(1\text{-TD}) &\leq \max_{0 \leq k \leq n} (m(k) - k) \\
&\leq \max_{0 \leq k \leq n} \left\{ \min\left\{\frac{3k}{2} + 1, n - k\right\} - k \right\} \\
&= \max_{0 \leq k \leq n} \left\{ \min\left\{\frac{k}{2} + 1, n - 2k\right\} \right\} \\
&= \frac{n + 4}{5},
\end{aligned} \tag{6}$$

where the last equation is achieved by setting  $\frac{k}{2} + 1$  equal to  $n - 2k$ , which implies  $k = \frac{2(n-1)}{5}$ . Finally by substituting (6) in Equation (1) we have

$$|M^*| \geq \frac{2(n-1)}{5}.$$

□

## 6 Blocking TD-Delaunay graphs

In this section we consider the problem of blocking TD-Delaunay graphs. Let  $P$  be a set of  $n$  points in the plane such that no pair of points of  $P$  is collinear in the  $l^0$ ,  $l^{60}$ , and  $l^{120}$  directions. Recall that a point set  $K$  blocks  $k$ -TD( $P$ ) if in  $k$ -TD( $P \cup K$ ) there is no edge connecting two points in  $P$ . That is,  $P$  is an independent set in  $k$ -TD( $P \cup K$ ).

**Theorem 9.** *At least  $\lceil \frac{(k+1)(n-1)}{3} \rceil$  points are necessary to block  $k$ -TD( $P$ ).*



*Proof.* Let  $K$  be a set of  $m$  points which blocks  $k$ -TD( $P$ ). Let  $G(\mathcal{P})$  be a complete graph with vertex set  $\mathcal{P} = P$ . Let  $\mathcal{T}$  be a minimum spanning tree of  $G(\mathcal{P})$  and let  $T$  be the set of triangles corresponding to the edges of  $\mathcal{T}$ . It is obvious that  $|T| = n - 1$ . By Lemma 2 the triangles in  $T$  are empty, thus, the edges of  $\mathcal{T}$  belong to any  $k$ -TD( $P$ ) where  $k \geq 0$ . To block each edge, corresponding to a triangle in  $T$ , at least  $k + 1$  points are necessary. By Lemma 3 each point in  $K$  can lie in at most three triangles of  $T$ . Therefore,  $m \geq \lceil \frac{(k+1)(n-1)}{3} \rceil$ , which implies that at least  $\lceil \frac{(k+1)(n-1)}{3} \rceil$  points are necessary to block all the edges of  $\mathcal{T}$  and hence  $k$ -TD( $P$ ).  $\square$

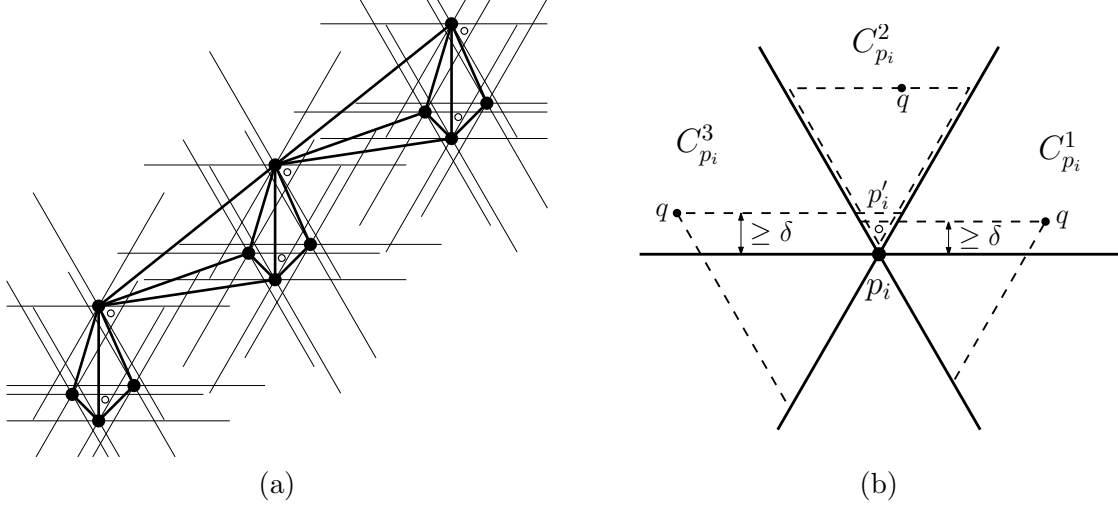


Figure 12: (a) a 0-TD graph which is shown in bold edges is blocked by  $\lceil \frac{n-1}{2} \rceil$  white points, (b)  $p'_i$  blocks all the edges connecting  $p_i$  to the vertices above  $l_{p_i}^0$ .

Theorem 9 gives a lower bound on the number of points that are necessary to block a TD-Delaunay graph. By this theorem, at least  $\lceil \frac{n-1}{3} \rceil$ ,  $\lceil \frac{2(n-1)}{3} \rceil$ ,  $n - 1$  points are necessary to block 0-, 1-, 2-TD( $P$ ) respectively. Now we introduce another formula which gives a better lower bound for 0-TD. For a point set  $P$ , let  $\nu_k(P)$  and  $\alpha_k(P)$  respectively denote the size of a maximum matching and a maximum independent set in  $k$ -TD( $P$ ). For every edge in the maximum matching, at most one of its endpoints can be in the maximum independent set. Thus,

$$\alpha_k(P) \leq |P| - \nu_k(P). \quad (7)$$

Let  $K$  be a set of  $m$  points which blocks  $k$ -TD( $P$ ). By definition there is no edge between points of  $P$  in  $k$ -TD( $P \cup K$ ). That is,  $P$  is an independent set in  $k$ -TD( $P \cup K$ ). Thus,

$$n \leq \alpha_k(P \cup K). \quad (8)$$

By (7) and (8) we have

$$n \leq \alpha_k(P \cup K) \leq (n + m) - \nu_k(P \cup K). \quad (9)$$

**Theorem 10.** At least  $\lceil \frac{n-1}{2} \rceil$  points are necessary to block 0-TD( $P$ ).

*Proof.* Let  $K$  be a set of  $m$  points which blocks  $k$ -TD( $P$ ). Consider 0-TD( $P \cup K$ ). It is known that the  $\nu_0(P \cup K) \geq \lceil \frac{n+m-1}{3} \rceil$ ; see [4]. By Inequality (9),

$$n \leq (n + m) - \lceil \frac{n + m - 1}{3} \rceil \leq \frac{2(n + m) + 1}{3},$$

and consequently  $m \geq \lceil \frac{n-1}{2} \rceil$  (note that  $m$  is an integer number).  $\square$

Figure 12(a) shows a 0-TD graph on a set of 12 points which is blocked by 6 points. By removing the topmost point we obtain a set with odd number of points which can be blocked by 5 points. Thus, the lower bound provided by Theorem 10 is tight.

Now let  $k = 1$ . By Theorem 8 we have  $\nu_1(P \cup K) \geq \frac{2((n+m)-1)}{5}$ , and by Inequality (9)

$$n \leq (n+m) - \frac{2((n+m)-1)}{5} = \frac{3(n+m)+2}{5},$$

and consequently  $m \geq \lceil \frac{2(n-1)}{3} \rceil$ ; the same lower bound as in Theorem 9.

Now let  $k = 2$ . By Theorem 7 we have  $\nu_2(P \cup K) = \lfloor \frac{n+m}{2} \rfloor$  (note that  $n+m$  may be odd). By Inequality (9)

$$n \leq (n+m) - \lfloor \frac{n+m}{2} \rfloor = \lceil \frac{n+m}{2} \rceil,$$

and consequently  $m \geq n$ , where  $n+m$  is even, and  $m \geq n-1$ , where  $n+m$  is odd.

**Theorem 11.** *There exists a set  $K$  of  $n-1$  points that blocks 0-TD( $P$ ).*

*Proof.* Let  $d^0(p, q)$  be the Euclidean distance between  $l_p^0$  and  $l_q^0$ . Let  $\delta = \min\{d^0(p, q) : p, q \in P\}$ . For each point  $p \in P$  let  $p(x)$  and  $p(y)$  respectively denote the  $x$  and  $y$  coordinates of  $p$  in the plane. Let  $p_1, \dots, p_n$  be the points of  $P$  in the increasing order of their  $y$ -coordinate. Let  $K = \{p'_i : p'_i(x) = p_i(x), p'_i(y) = p_i(y) + \epsilon, \epsilon < \delta, 1 \leq i \leq n-1\}$ . See Figure 12(b). For each point  $p_i$ , let  $E_{p_i}$  (resp.  $\overline{E}_{p_i}$ ) denote the edges of 0-TD( $P$ ) between  $p_i$  and the points above  $l_{p_i}^0$  (resp. below  $l_{p_i}^0$ ). It is easy to see that the downward triangle between  $p_i$  and any point  $q$  above  $l_{p_i}^0$  (i.e. any point  $q \in C_{p_i}^1 \cup C_{p_i}^2 \cup C_{p_i}^3$ ) contains  $p'_i$ . Thus,  $p'_i$  blocks all the edges in  $E_{p_i}$ . In addition, the edges in  $\overline{E}_{p_i}$  are blocked by  $p'_1, \dots, p'_{i-1}$ . Therefore, all the edges of 0-TD( $P$ ) are blocked by the  $n-1$  points in  $K$ .  $\square$

Note that the bound of Theorem 11 is tight, because 0-TD( $P$ ) can be a path representing  $n-1$  disjoint triangles and for each triangle we need at least one point to block its corresponding edge. We can extend the result of Theorem 11 to  $k$ -TD( $P$ ) where  $k \geq 1$ . For each point  $p_i$  we put  $k+1$  copies of  $p'_i$  very close to  $p_i$ . Thus,

**Corollary 3.** *There exists a set  $K$  of  $(k+1)(n-1)$  points that blocks  $k$ -TD( $P$ ).*

## 7 Conclusion

In this paper, we considered some combinatorial properties of higher-order triangular-distance Delaunay graphs of a point set  $P$ . We proved that

- $k$ -TD is  $(k+1)$  connected.
- 2-TD contains a bottleneck biconnected spanning graph of  $P$ .
- 7-TD contains a bottleneck Hamiltonian cycle and 5-TD may not have any.
- 6-TD contains a bottleneck perfect matching and 5-TD may not have any.
- 1-TD has a matching of size at least  $\frac{2(n-1)}{5}$ .
- 2-TD has a perfect matching when  $P$  has an even number of points.
- $\lceil \frac{n-1}{2} \rceil$  points are necessary to block 0-TD.
- $\lceil \frac{(k+1)(n-1)}{3} \rceil$  points are necessary and  $(k+1)(n-1)$  points are sufficient to block  $k$ -TD.

We leave a number of open problems:

- What is a tight lower bound for the size of maximum matching in 1-TD?
- Does 6-TD contain a bottleneck Hamiltonian cycle?
- As shown in Figure 1(a) 0-TD may not have a Hamiltonian cycle. For which values of  $k = 1, \dots, 6$ , is the graph  $k$ -TD Hamiltonian?

## References

- [1] M. Abellanas, P. Bose, J. García-López, F. Hurtado, C. M. Nicolás, and P. Ramos. On structural and graph theoretic properties of higher order Delaunay graphs. *Int. J. Comput. Geometry Appl.*, 19(6):595–615, 2009.
- [2] O. Aichholzer, R. F. Monroy, T. Hackl, M. J. van Kreveld, A. Pilz, P. Ramos, and B. Vogtenhuber. Blocking Delaunay triangulations. *Comput. Geom.*, 46(2):154–159, 2013.
- [3] B. Aronov, M. Dulieu, and F. Hurtado. Witness Gabriel graphs. *Comput. Geom.*, 46(7):894–908, 2013.
- [4] J. Babu, A. Biniarz, A. Maheshwari, and M. Smid. Fixed-orientation equilateral triangle matching of point sets. To appear in *Theoretical Computer Science*.
- [5] C. Berge. Sur le couplage maximum d’un graphe. *C. R. Acad. Sci. Paris*, 247:258–259, 1958.
- [6] N. Bonichon, C. Gavoille, N. Hanusse, and D. Ilcinkas. Connections between theta-graphs, Delaunay triangulations, and orthogonal surfaces. In *WG*, pages 266–278, 2010.
- [7] P. Bose, P. Carmi, S. Collette, and M. H. M. Smid. On the stretch factor of convex Delaunay graphs. *Journal of Computational Geometry*, 1(1):41–56, 2010.
- [8] P. Bose, S. Collette, F. Hurtado, M. Korman, S. Langerman, V. Sacristan, and M. Saumell. Some properties of  $k$ -Delaunay and  $k$ -Gabriel graphs. *Comput. Geom.*, 46(2):131–139, 2013.
- [9] M.-S. Chang, C. Y. Tang, and R. C. T. Lee. 20-relative neighborhood graphs are Hamiltonian. *Journal of Graph Theory*, 15(5):543–557, 1991.
- [10] M.-S. Chang, C. Y. Tang, and R. C. T. Lee. Solving the Euclidean bottleneck biconnected edge subgraph problem by 2-relative neighborhood graphs. *Discrete Applied Mathematics*, 39(1):1–12, 1992.
- [11] M.-S. Chang, C. Y. Tang, and R. C. T. Lee. Solving the Euclidean bottleneck matching problem by  $k$ -relative neighborhood graphs. *Algorithmica*, 8(3):177–194, 1992.
- [12] P. Chew. There are planar graphs almost as good as the complete graph. *J. Comput. Syst. Sci.*, 39(2):205–219, 1989.
- [13] M. B. Dillencourt. A non-hamiltonian, nondegenerate Delaunay triangulation. *Inf. Process. Lett.*, 25(3):149–151, 1987.
- [14] M. B. Dillencourt. Toughness and Delaunay triangulations. *Discrete & Computational Geometry*, 5:575–601, 1990.

- [15] T. Lukovszki. New results of fault tolerant geometric spanners. In *WADS*, pages 193–204, 1999.
- [16] W. T. Tutte. The factorization of linear graphs. *Journal of the London Mathematical Society*, 22(2):107–111, 1947.